Comment on 'On the general solution for the modified Emden-type equation $\ddot{x} + \alpha x \dot{x} + \beta x^3 = 0$'

2008 J. Phys. A: Math. Theor. 41 068001

(http://iopscience.iop.org/1751-8121/41/6/068001)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 068001 (4pp) doi:10.1088/1751-8113/41/6/068001

COMMENT

Comment on 'On the general solution for the modified Emden-type equation $\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0$'

R Iacono

ENEA, CR Casaccia, Rome, Italy

Received 15 May 2007, in final form 26 October 2007
Published 29 January 2008
Online at stacks.iop.org/JPhysA/41/068001

Abstract
It is shown that the modified Emden-type equation in the title belongs to a class of Liènard's equations that may be mapped into separable Abel's equations, and are therefore integrable. First integrals for equations of this class are easily derived (there are three of them, corresponding to three distinct dynamical regimes), and integrated, to yield parametric representations of the corresponding solutions.

PACS numbers: 02.30.Hq, 02.30.Ik, 05.45.−a

Recent work by Chandrasekar *et al* [1] (CH07 herafter) has focused on the modified Emden-type equation

$$\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0, \tag{1}$$

a special case of the Liénard equation (LE hereafter),

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{2}$$

of interest for the applications. In previous work [2], the authors of CH07 had shown integrability of (1) for $\alpha^2 \geqslant 8\beta$, without giving the corresponding solutions. In CH07, they complete their analysis of (1), proving integrability for $\alpha^2 < 8\beta$ and constructing solutions over the whole parameter space. In order to do so, they use a method by Duarte *et al* [3]—which requires the solution of a system of nonlinear pde's—to obtain first integrals not explicitly dependent on time, for the three cases $\alpha^2 < 8\beta$, $\alpha^2 = 8\beta$ and $\alpha^2 > 8\beta$. Then, judging the first integrals very difficult to integrate directly, they use appropriate Hamiltonian formulations to derive the explicit solution for $\alpha^2 = 8\beta$, and implicit solutions for the other two cases.

Here, we note that a complete characterization of the solutions of (1) can be obtained more easily by exploiting the basic connection between the LE and the Abel equation (AE herafter). Following this route naturally leads to the three distinct dynamical regimes and to the corresponding first integrals, and allows one to integrate the first integrals in a straightforward manner. In the following, we shall use this approach to construct a parametric solution for a whole class of LE's, that includes (1), and other interesting LE's with polynomial coefficients, as special cases.

It is well known that the transformation

$$\dot{x} \equiv \frac{dx}{dt} = \xi(x) \tag{3}$$

maps (2) into the AE of the second kind

$$\xi\xi' + f(x)\xi + g(x) = 0, \qquad ' \equiv \frac{d}{dx}, \tag{4}$$

which has been widely investigated. A general condition for the separability of (4) was given by Liouville [4]. A special case of Liouville's condition, of interest for our problem, is

$$\left(\frac{g}{f}\right)' = \delta f, \tag{5}$$

with δ an arbitrary constant. When (5) is satisfied, the change of dependent variable

$$\xi = \frac{g}{f}\frac{1}{w} \tag{6}$$

yields the separable equation

$$w' = \frac{f^2}{g} w(w^2 + w + \delta). \tag{7}$$

Thus, (5) identifies a class of LE's that are mapped by (3) into separable AE's, and are therefore integrable. Clearly, (1) belongs to this class, since its coefficients satisfy (5), with $\delta = 2\beta/\alpha^2$.

The solution for this class of equations may be easily computed. Separating dependences from w and x, making further use of (5), and integrating on both sides of (7), gives

$$I(w) \equiv \int \frac{dw}{w(w^2 + w + \delta)} = \frac{1}{\delta}\int dx \frac{f}{\int dx\, f} = \frac{1}{\delta}\int d\left(\log \int dx\, f\right). \tag{8}$$

The integral $I(w)$ is given by (see, e.g., [5])

$$I(w) = 4\left[\frac{1}{2w+1} - \log\frac{2w+1}{w}\right], \qquad \Delta = 0, \tag{9}$$

$$I(w) = \frac{1}{2\delta}\left[\log w^2 - \log W - \frac{1}{\sqrt{\Delta}}\log\frac{2w+1-\sqrt{\Delta}}{2w+1+\sqrt{\Delta}}\right], \qquad \Delta > 0, \tag{10}$$

$$I(w) = \frac{1}{2\delta}\left[\log w^2 - \log W - \frac{2}{\sqrt{-\Delta}}\tan^{-1}\left(\frac{2w+1}{\sqrt{-\Delta}}\right)\right], \qquad \Delta < 0, \tag{11}$$

where

$$W \equiv w^2 + w + \delta, \qquad \Delta \equiv 1 - 4\delta. \tag{12}$$

Thus, three different dynamical regimes are naturally obtained, corresponding to the cases in which the discriminant Δ of the quadratic polynomial W is vanishing, positive, or negative, respectively. In the case of (1), this translates into $\alpha^2 = 8\beta$, $\alpha^2 > 8\beta$ and $\alpha^2 < \beta$, the three regimes of CH07.

If we now recall that

$$w = \frac{g(x)}{f(x)}\frac{1}{\dot{x}}, \tag{13}$$

and note that the integration in (8) involves an arbitrary constant, it becomes clear that (8) is none other than a first integral for the class of LE's in consideration (in fact, three of them, for the three different regimes). In the case of (1), (13) takes the form

$$w = \frac{\beta}{\alpha}\frac{x^2}{\dot{x}}, \tag{14}$$

and the rhs of (8) yields

$$\frac{\alpha^2}{\beta}\log x + C, \tag{15}$$

with C an arbitrary constant. Replacing (14) in (9)–(11), and the resulting expressions, together with (15), in (8), immediately yields the first integrals (2)–(4) of CH07.

At this point, one can either use a Hamiltonian approach, as in CH07, or directly construct an implicit solution as follows. Once $I(w)$ is computed, integration of the rhs of (8) gives an expression of $\int \mathrm{d}x\, f(x)$ in terms of w and of an arbitrary constant C. If $\int \mathrm{d}x\, f(x)$ is an invertible function of x, one can then compute

$$x = x(w, C). \tag{16}$$

Replacing this expression in (3), one gets

$$\frac{1}{\delta}\frac{w}{\int \mathrm{d}x\, f}\frac{\mathrm{d}x}{\mathrm{d}w}(w, C)\,\mathrm{d}w = \mathrm{d}t, \tag{17}$$

which, upon integration, yields

$$t - t_0 = \frac{1}{\delta}\int \mathrm{d}w \frac{w}{\int \mathrm{d}x\, f}\frac{\mathrm{d}x}{\mathrm{d}w}, \tag{18}$$

with t_0 the second integration constant. Together with (16), (18) provides a parametric representation of the general integral of the class of LE's with coefficients satisfying (5).

An interesting equation that belongs to this class is a natural generalization of (1),

$$\ddot{x} + \alpha x^q \dot{x} + \beta x^{2q+1} = 0, \tag{19}$$

with q a real number. Integrability of special cases of (19) has been discussed in [2] and [6]. But since (19) belongs to the class (5), with $\delta = (q+1)\beta/\alpha^2$, it is in fact integrable for arbitrary values of the parameters. The three dynamical regimes for this equation are determined by the relative sizes of α^2 and $4(q+1)\beta$. Integrating (8) gives

$$x = x_0 \exp\left[\frac{\beta}{\alpha^2} I(w)\right], \tag{20}$$

with x_0 an integration constant, and replacing this result in (18) we find

$$t - t_0 = \frac{x_0^{-q}}{\alpha}\int \mathrm{d}w \frac{1}{W(w)} \exp\left(-\frac{q\beta}{\alpha^2} I(w)\right). \tag{21}$$

Once the corresponding forms of $W(w)$ and of $I(w)$ are inserted, (21) yields a parametric representation of the general integral of (19), for arbitrary values of the parameters α, β and q.

Finally, we note that the transformation

$$x = \mathrm{e}^{\lambda\tau} v, \qquad t = -\frac{1}{\gamma}\,\mathrm{e}^{-\gamma\tau}; \qquad \gamma = q\lambda, \tag{22}$$

maps (19) into

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\tau^2} + [\alpha v^q + (q+2)\lambda]\frac{\mathrm{d}v}{\mathrm{d}\tau} + \beta v^{2q+1} + \alpha\lambda v^{q+1} + \lambda^2 (q+1) v = 0, \tag{23}$$

which provides a direct generalization of one of the integrable equations obtained in [2] (equation (16)). Another integrable case [2, equation (12)] is obtained from (23) by taking

$$\beta = 0, \qquad \alpha = k_1, \qquad \lambda = k_2. \tag{24}$$

Equation (15) of [2] also has coefficients satisfying (5), and is therefore integrable. However, $\int f(x)$ cannot be inverted, in general, for the latter equations, and explicit expressions of the integral (18) cannot be derived analytically.

To summarize, we can say that the main purpose of this comment is to draw attention to the basic, fruitful connection between the LE and the AE, sometimes neglected in recent literature. We do not mean to be critical about the application of new methods, such as that introduced in [3]; we only wish to underline that before using a more complicated approach, it is useful to know exactly what old methods, such as those here reminded, may provide. Besides, the connection between the LE and the AE could also be used in the other direction: new integrable cases of the LE, obtained, for example, by using the methods employed in [1, 2], could provide new, nontrivial solvable cases of the AE.

References

[1] Chandrasekar V K, Senthilvelan M and Lakshmanan M 2007 *J. Phys. A: Math. Gen.* **40** 4717

[2] Chandrasekar V K, Pandey S N, Senthilvelan M and Lakshmanan M 2006 *J. Math. Phys.* **47** 023508

[3] Duarte L G S, Duarte S E S, da Mota A C P and Skea J E F 2001 *J. Phys. A: Math. Gen.* **34** 3015

[4] Liouville R 1902 *Acta Math.* **26** 55
See also Liouville R 1887 *C. R.* **103** 460

[5] Beyer W H 1987 *CRC Handbook of Mathematical Sciences* 6th edn (Boca Raton: CRC Press)

[6] Feix M R, Geronimi C, Cairo L, Leach P G L, Lemmer R L and Bouquet S 1997 *J. Phys. A: Math. Gen.* **30** 7437